Lefschetz-Pontrjagin Duality for Differential Characters

by

Reese Harvey and Blaine Lawson[1]

Abstract

A theory of differential characters is developed for manifolds with boundary. This is done from both the Cheeger-Simons and the deRham-Federer viewpoints. The central result of the paper is the formulation and proof of a Lefschetz-Pontrjagin Duality Theorem, which asserts that the pairing

$$\widehat{I\!\!H}^k(X, \partial X) \times \widehat{I\!\!H}^{n-k-1}(X) \;\longrightarrow\; S^1$$

given by

$$(\alpha, \beta) \;\mapsto\; (\alpha * \beta)\,[X]$$

induces isomorphisms

$$\mathcal{D} : \widehat{I\!\!H}^k(X, \partial X) \to \mathrm{Hom}_\infty(\widehat{I\!\!H}^{n-k-1}(X),\, S^1)$$
$$\mathcal{D}' : \widehat{I\!\!H}^{n-k-1}(X) \to \mathrm{Hom}_\infty(\widehat{I\!\!H}^k(X, \partial X),\, S^1)$$

onto the smooth Pontrjagin duals. In particular, $\mathcal{D}$ and $\mathcal{D}'$ are injective with dense range in the group of all continuous homomorphisms into the circle. A coboundary map is introduced which yields a long sequence for the character groups associated to the pair $(X, \partial X)$. The relation of the sequence to the duality mappings is analyzed.

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§0. Introduction.

The theory of differential characters, introduced by Jim Simons and Jeff Cheeger in 1973, is of basic importance in geometry. It provides a wealth of invariants for bundles with connection starting with the classical one of Chern-Simons in dimension 3 and including large families of invariants for flat bundles and foliations. Its cardinal property is that it forms the natural receiving space for a refined Chern-Weil theory. This theory subsumes integral characteristic classes and the classical Chern-Weil characteristic forms. It also tracks certain "transgression terms" which give cohomologies between smooth and singular cocycles and lead to interesting secondary invariants.

Each standard characteristic classes has a refinement in the group of differential characters. Thus for a complex bundle with unitary connection, refined Chern classes $\widehat{c}_k$ are defined and the total class

$$\widehat{c} = 1 + \widehat{c}_1 + \widehat{c}_2 + \cdots : I\!K(X) \longrightarrow \widehat{I\!H}^*(X)$$

gives a natural transformation from the K-theory of bundles *with connection* to differential characters which satisfies the Whitney sum formula: $\widehat{c}(E \oplus F) = \widehat{c}(E) * \widehat{c}(F)$. This last property leads to non-conformal immersion theorems in riemannian geometry.

Differential characters form a highly structured theory with certain aspects of cohomology: contravariant functoriality, ring structure, and a pairing to cycles. There are deRham-Federer formulations of the theory [GS$_1$], [H], [HLZ], analogous to those given for cohomology, which are useful for example in the theory of singular connections [HL$_1$],[HL$_2$]. Furthermore, the groups $\widehat{I\!H}^k(X)$ of differential characters carry a natural topology. The connected component of 0 in this group consists of the **smooth characters**, those which can be represented by smooth differential forms.

In [HLZ], where the deRham-Federer appoach is developed in detail, the authors showed that differential characters satisfy **Poincaré-Pontjagin duality**: On an oriented n dimensional manifold X the pairing

$$\widehat{I\!H}^k(X) \times \widehat{I\!H}^{n-k-1}_{\rm cpt}(X) \;\longrightarrow\; S^1$$

given by

$$(\alpha, \beta) \;\mapsto\; (\alpha * \beta)[X]$$

(where $\widehat{I\!H}^*_{\mathrm{cpt}}$ denotes characters with compact support) induces injective maps

$$\widehat{I\!H}^k(X) \;\to\; \mathrm{Hom}\left(\widehat{I\!H}^{n-k-1}_{\mathrm{cpt}}(X), S^1\right) \qquad \text{and} \qquad \widehat{I\!H}^{n-k-1}_{\mathrm{cpt}}(X) \;\to\; \mathrm{Hom}\left(\widehat{I\!H}^k(X), S^1\right)$$

with dense range in the groups of continuous homomorphisms into the circle. Moreover this range consists exactly of the **smooth** homomorphisms. These are defined precisely in §4 but can be thought of roughly as follows. The connected component of 0 in $\widehat{I\!H}^k(X)$ consists essentially (i.e., up to a finite-dimensional torus factor) of the exact $(k+1)$-forms $d\mathcal{E}^{k+1}(X)$ with the C^∞-topology. Now $\mathrm{Hom}(d\mathcal{E}^{k+1}(X), S^1) = \mathrm{Hom}(d\mathcal{E}^{k+1}(X), \mathbb{R})$ is just the vector space dual. This is simply a quotient of the space of *currents*, the $(n-k-1)$-forms with distribution coefficients. The smooth dual corresponds to those forms which have smooth coefficients.

In this paper we formulate the theory of differential characters for compact manifolds with boundary $(X, \partial X)$ and prove a Lefschetz-Pontrjagin Duality Theorem analogous to the one above. To do this we introduce the relative groups $\widehat{I\!H}^*(X, \partial X)$ and develop the theory from [HLZ] for this case. The main theorem asserts the existence of a pairing

$$\widehat{I\!H}^k(X) \times \widehat{I\!H}^{n-k-1}(X, \partial X) \;\longrightarrow\; S^1$$

given by $(\alpha, \beta) \;\mapsto\; (\alpha * \beta)[X]$ and inducing injective maps with dense range as above.

The two pairings above have a formal similarity but are far from the same. The delicate part of these dualities comes from the differential form component of characters. In the first pairing (on possibly non-compact manifolds) we contrast forms having no growth restrictions at infinity with forms with compact support. The second dualtiy (on compact manfiolds with boundary) opposes forms smooth up to the bounday with forms which restrict to zero on the boundary.

In cohomology theory there are long exact sequences for the pair $(X, \partial X)$ which interlace the Pontrjagin and Lefschetz Duality mappings. In the last sections of this paper the parallel structure for differential characters is studied. We introduce coboundary maps $\partial : \widehat{I\!H}^k(X) \to \widehat{I\!H}^{k+1}(X, \partial X)$, yielding long sequences which intertwine the duality mappings and reduce to the standard picture under the natural transformation to integral cohomology.

§**1. Differential characters on manifolds with boundary.** Let X be a compact oriented differentiable n-manifold with boundary ∂X. Let $\mathcal{E}^*(X)$ denote the de Rham complex of differential forms which are smooth up to the boundary, and set

$$\mathcal{E}^*(X, \partial X) \;=\; \{\phi \in \mathcal{E}^*(X) \;:\; \phi\big|_{\partial X} = 0\}.$$

The cohomology of this complex is naturally isomorphic to $H^*(X, \partial X; \mathbb{R})$. Let $C_*(X)$ denote the complex of C^∞-singular chains on X and $C_*(X, \partial X) \;\equiv\; C_*(X)/C_*(\partial X)$ the relative complex. Denote by

$$Z_*(X, \partial X) \;\equiv\; \{c \in C_*(X, \partial X) \;:\; \partial c = 0\}$$

the cycles in this complex. We begin with definitions of differential characters in the spirit of Cheeger-Simons.

Definition 1.1. The group of **differential characters** of degree k on X is the set of homomorphisms

$$\hat{H}^k(X; \mathbb{R}/\mathbb{Z}) \;\equiv\; \{\alpha \in \mathrm{Hom}(Z_k(X), S^1) \;:\; \delta(\alpha) \in \mathcal{E}^{k+1}(X)\}$$

where δ denotes the coboundary. Similarly the group of **relative differential characters** of degree k on $(X, \partial X)$ is defined to be

$$\hat{H}^k(X, \partial X; \mathbb{R}/\mathbb{Z}) \;\equiv\; \{\alpha \in \mathrm{Hom}(Z_k(X, \partial X), S^1) \;:\; \delta(\alpha) \in \mathcal{E}^{k+1}(X, \partial X)\}$$

Inclusion and restriction give maps $\widehat{I\!H}^k(X, \partial X) \xrightarrow{j} \widehat{I\!H}^k(X) \xrightarrow{\rho} \widehat{I\!H}^k(\partial X)$. with $\rho \circ j = 0$.

There is an alternative de Rham-Federer approach to these groups. Set

$$\begin{aligned}
\mathcal{E}^k_{L^1_{\rm loc}}(X) \;&\equiv\; k\text{-forms on } X \text{ with } L^1_{\rm loc}\text{-coefficients} \\
\mathcal{R}^k(X) \;&\equiv\; \text{the rectifiable currents of degree } k \text{ (dimension } n-k) \text{ on } X \\
\mathcal{E}^k_{L^1_{\rm loc}}(X, \partial X) \;&\equiv\; \{a \in \mathcal{E}^k_{L^1_{\rm loc}}(X) \;:\; a \text{ in smooth in a neighborhood of } \partial X \text{ and } a\big|_{\partial X} = 0\} \\
\mathcal{R}^k_{\rm cpt}(X - \partial X) \;&\equiv\; \{R \in \mathcal{R}^k(X) \;:\; \mathrm{supp}(R) \subset X - \partial X\}
\end{aligned}$$

Definition 1.2. An element $a \in \mathcal{E}^k_{L^1_{\rm loc}}(X)$ is called a **spark** of degree k on X if

$$da \;=\; \phi - R \qquad \text{where } \phi \in \mathcal{E}^{k+1}(X) \text{ and } R \in \mathcal{R}^{k+1}(X). \tag{1.3}$$

Denote by $\mathcal{S}^k(X)$ the group of all such sparks and by $\mathcal{T}^k(X)$ the subgroup of all $a \in \mathcal{S}^k(X)$ such that $a = db + S$ where $b \in \mathcal{E}^{k-1}_{L^1_{\rm loc}}(X)$ $S \in \mathcal{R}^k_{\rm cpt}(X)$. Then the group of de Rham-Federer characters of degree k on X is defined to be the quotient

$$\widehat{I\!H}^k(X) \;\equiv\; \mathcal{S}^k(X)/\mathcal{T}^k(X).$$

We define **relative sparks** and **relative de Rham-Federer characters** on $(X, \partial X)$ by

$$\begin{aligned}
\mathcal{S}^k(X, \partial X) &\equiv \{a \in \mathcal{E}^k_{L^1_{\rm loc}}(X, \partial X) \;:\; da = \phi - r, \;\; \phi \in \mathcal{E}^{k+1}(X, \partial X) \text{ and } R \in \mathcal{R}^{k+1}_{\rm cpt}(X - \partial X)\} \\
\mathcal{T}^k(X, \partial X) &\equiv \{a \in \mathcal{S}^k(X, \partial X) \;:\; a = db + S, \;\; b \in \mathcal{E}^{k-1}_{L^1_{\rm loc}}(X, \partial X) \text{ and } S \in \mathcal{R}^k_{\rm cpt}(X - \partial X)\} \\
\widehat{I\!H}^k(X, \partial X) \;&\equiv\; \mathcal{S}^k(X, \partial X)/\mathcal{T}^k(X, \partial X).
\end{aligned}$$

The decomposition (1.3) is unique. In fact we have the following. Recall that a current T is said to be **integrally flat** if it can be written as $T = R + dS$ where R and S are rectifiable. Then from [HLZ; 1.5] one concludes:

Proposition 1.4. *Let a be any current of degree k on X such that $da = \phi - R$ where $\phi \in \mathcal{E}^{k+1}(X)$ and R is integrally flat. If $da = \phi' - R'$ is a similar decomposition, then $\phi = \phi'$ and $R = R'$. Furthermore,*

$$d\phi \ = \ 0 \qquad \text{and} \qquad dR\big|_{X-\partial X} \ = \ 0$$

and ϕ has integral periods on cycles in X. In the case that $\phi \in \mathcal{E}^{k+1}(X, \partial X)$ and supp$(R) \subset X - \partial X$, one has that $dR = 0$ and ϕ has integral periods on all relative cycles in $(X, \partial X)$.

Set

$$\begin{aligned}
\mathcal{Z}_0^\ell(X) &= \{\phi \in \mathcal{E}^\ell(X) : d\phi = 0 \text{ and } \phi \text{ has integral periods}\} \\
\mathcal{Z}_0^\ell(X, \partial X) &= \{\phi \in \mathcal{E}^\ell(X, \partial X) : d\phi = 0 \text{ and } \phi \text{ has} \\
&\qquad\qquad \text{integral periods on relative cycles in } (X, \partial X)\ \} \\
Z_{\mathrm{rect}}^\ell(X) &= \{R \in R^\ell(X) : dR\big|_{X-\partial X} = 0\} \\
Z_{\mathrm{rect}}^\ell(X, \partial X) &= \{R \in R_{\mathrm{cpt}}^\ell(X - \partial X) : dR = 0\}
\end{aligned} \tag{1.5}$$

Corollary 1.6. *Taking $d_1 a = \phi$ and $d_2 a = R$ from the decomposition (1.3) gives well-defined mappings*

$$d_1 : S^k(X) \ \longrightarrow\ \mathcal{Z}_0^{k+1}(X), \qquad d_2 : S^k(X) \ \longrightarrow\ Z_{\mathrm{rect}}^{k+1}(X), \qquad \text{and}$$

$$d_1 : S^k(X, \partial X) \ \longrightarrow\ \mathcal{Z}_0^{k+1}(X, \partial X), \qquad d_2 : S^k(X, \partial X) \ \longrightarrow\ Z_{\mathrm{rect}}^{k+1}(X - \partial X)$$

Proposition 1.7. *There are natural isomorphisms*

$$\Psi : \widehat{I\!\!H}^k(X) \xrightarrow{\cong} \hat{H}^k(X; \mathbb{R}/\mathbb{Z}) \qquad \text{and} \qquad \Psi : \widehat{I\!\!H}^k(X, \partial X) \xrightarrow{\cong} \hat{H}^k(X, \partial X; \mathbb{R}/\mathbb{Z})$$

induced by integration.

Proof. The first is proved in [HLZ]. The argument for the second is exactly the same. $\square$

Remark 1.8. In [HLZ] we showed that there are many different (but equivalent) deRham-Federer definitions of differential characters on a manifold Y. Each of these different presentations has obvious analogues for $\widehat{I\!\!H}^*(X)$ and $\widehat{I\!\!H}^*(X, \partial X)$. The proof of the equivlance of these definitions closely follows the arguments in [HLZ, §2] and will not be given here. However, this flexibility in definitions is important in our treatment of the $*$-product.

To illustrate the point we give one example. Recall that a current R on X is called **integrally flat** if $R = S + dT$ where S and T are rectifiable. Denote by $\mathcal{D}'^k(X) \equiv \{\mathcal{E}^{n-k}(X)\}'$ the space of currents of degree k on X. Let $\mathcal{S}_{\max}^k(X, \partial X)$ denote the set

of $a \in \mathcal{D}'^k(X)$ such that a is smooth near ∂X, $a\big|_{\partial X} = 0$, and $da = \phi - R$ where $\phi \in \mathcal{E}^{k+1}(X, \partial X)$ and R is integrally flat. Let $\mathcal{T}^k_{\max}(X, \partial X)$ denote the subgroup of elements of the form $db + S$ where b is smooth near ∂X, $b\big|_{\partial X} = 0$, and S is integrally flat. Then the inclusion $\mathcal{S}^k(X, \partial X) \subset \mathcal{S}^k_{\max}(X, \partial X)$ induces an isomorphism

$$\widehat{I\!H}^k(X, \partial X) \;\cong\; \mathcal{S}^k_{\max}(X, \partial X)/\mathcal{T}^k_{\max}(X, \partial X)$$

§**2. The exact sequences.** The fundamental exact sequences established by Cheeger and Simons in [CS] carry over to the relative case.

Definition 2.1. A character $\alpha \in \widehat{I\!H}^k(X, \partial X)$ is said to be **smooth** if $\alpha = \langle a \rangle$ for a smooth form $a \in \mathcal{E}^k(X, \partial X)$. The set of smooth characters is denoted $\widehat{I\!H}^k_\infty(X, \partial X)$. There is a natural isomorphism

$$\widehat{I\!H}^k_\infty(X, \partial X) \;\cong\; \mathcal{E}^k(X, \partial X)/\mathcal{Z}^k_0(X, \partial X)$$

Proposition 2.2. *The mappings d_1 and d_2 induce functorial short exact sequences:*

$$\text{(A)} \qquad 0 \longrightarrow H^k(X, \partial X; S^1) \xrightarrow{j_1} \widehat{I\!H}^k(X, \partial X) \xrightarrow{\delta_1} \mathcal{Z}^{k+1}_0(X, \partial X) \longrightarrow 0,$$

$$\text{(B)} \qquad 0 \longrightarrow \widehat{I\!H}^k_\infty(X, \partial X) \xrightarrow{j_2} \widehat{I\!H}^k(X, \partial X) \xrightarrow{\delta_2} H^{k+1}(X, \partial X; \mathbb{Z}) \longrightarrow 0.$$

Proof. Note that ∂X has a cofinal system of tubular neighborhoods each of which is diffeomorphic to $\partial X \times [0, 1)$. We shall use the following elementary result.

Lemma 2.3. *For any $a \in \mathcal{E}^k(\partial X \times [0, 1))$ such that $da = 0$ and $a\big|_{\partial X} = 0$, there exists $b \in \mathcal{E}^{k-1}(\partial X \times [0, 1))$ such that $db = a$ and $b\big|_{\partial X} = 0$.*

Proof. Write $a = a_1 + dt \wedge a_2$ where a_1 and a_2 are forms on X whose coefficients depend smoothly on $t \in [0, 1)$, or in other words, $a_1(t)$, $a_2(t)$ are smooth curves in $\mathcal{E}^k(X)$ and $\mathcal{E}^{k-1}(X)$ repsectively with $a_1(0) = 0$. Now $da = d_x a_1 + dt \wedge \frac{\partial a_1}{\partial t} - dt \wedge d_x a_2 = 0$. We conclude that $d_x a_1 = 0$ and $d_x a_2 = \frac{\partial a_1}{\partial t}$. Since $a_1(0) = 0$ we have

$$a_1(t) \;=\; \int_0^t \frac{\partial a_1}{\partial t}(s)\, ds \;=\; \int_0^t d_x a_2(s)\, ds \;=\; d_x \int_0^t a_2(s)\, ds$$

Set $b \equiv \int_0^t a_2(s)\, ds$, and note that: $b\big|_{\partial X} = 0$, $d_x b = a_1$ and $\frac{\partial b}{\partial t} = a_2$. Hence, $a = db$. $\square$

We shall also need the following result. On any manifold Y let

$$\mathcal{F}^k(Y) \equiv \mathcal{E}^k_{L^1_{\mathrm{loc}}}(Y) + d\mathcal{E}^{k-1}_{L^1_{\mathrm{loc}}}(Y)$$

denote **flat currents** and $\mathcal{F}^k_{\mathrm{cpt}}(Y)$ those with compact support. Note that $d\mathcal{F}^k(Y) = d\mathcal{E}^k_{L^1_{\mathrm{loc}}}(Y)$. This definition of $\mathcal{F}^k_{\mathrm{cpt}}(Y)$ arises naturally in sheaf theory. However, the following equivalent definition will also be useful here.

Lemma 2.4. $\mathcal{F}^k_{\rm cpt}(Y) = \mathcal{E}^k_{L^1_{\rm loc},{\rm cpt}}(Y) + d\mathcal{E}^{k-1}_{L^1_{\rm loc},{\rm cpt}}(Y)$ and so $d\mathcal{F}^k_{\rm cpt}(Y) = d\mathcal{E}^k_{L^1_{\rm loc},{\rm cpt}}(Y)$

Proof. Fix $f \in \mathcal{F}^k_{\rm cpt}(Y)$ and write $f = a + db$ where $a \in \mathcal{E}^k_{L^1_{\rm loc}}(Y)$ and $b \in \mathcal{E}^{k-1}_{L^1_{\rm loc}}(Y)$. Let $K = \text{supp}\,(f)$, and note that in $N \equiv Y - K$ we have that $a = -db$. By standard de Rham theory there exists an $L^1_{\rm loc}$-form b_0 on N such that $a_\infty \equiv a + db_0$ is smooth on N. Furthermore since a_∞ is weakly exact on N there exists a smooth form b_∞ with $a_\infty = -db_\infty$ on N. Choose $\eta \in C^\infty_0(Y)$ with $\eta \equiv 1$ in a neighbothood of K, let $\chi = 1 - \eta$ and set $\widetilde{a} = a + d(\chi b_0 + \chi b_\infty)$ and $\widetilde{b} = b - \chi b_0 - \chi b_\infty$ with χ as above. Then $f = \widetilde{a} + d\widetilde{b}$ and $\widetilde{a}$ has compact support in Y.

Observe now that $f - \widetilde{a}$ is d-closed and has compact support in Y. Since $H^*(\mathcal{E}^*_{\rm cpt}(Y)) \cong H^*(\mathcal{F}^*_{\rm cpt}(Y))$ we conclude that there exist a smooth form ω and a flat form g, both having compact support on Y such that $f - \widetilde{a} = \omega + dg$. Now by the paragraph above we can write $g = b + de$ where b is $L^1_{\rm loc}$ with compact support. Hence $f = \widetilde{a} + \omega + db$. $\square$

We first prove the surjectivity of δ_1. Fix $\phi \in \mathcal{Z}^{k+1}_0(X, \partial X)$. Then by Lemma 2.3 there is a neighborhood $N \cong \partial X \times [0, 1)$ of ∂X and a form $A \in \mathcal{E}^k(N)$ with $dA = \phi$ and $A\big|_{\partial X} = 0$. Choose $\chi \in C^\infty_0(N)$ with $\chi \equiv 1$ in a neighborhood of ∂X, and set $\phi_0 = \phi - d(\chi A)$. Now $\text{supp}(\phi_0) \subset\subset X - \partial X$ and ϕ_0 has integral periods, so there exists a cycle $R \in Z^\ell_{\rm rect}(X, \partial X)$ with $[\phi_0 - R] = 0$ in $H^*_{\rm cpt}(X - \partial X; \mathbb{R})$. By Lemma 2.4 there are $L^1_{\rm loc}$-forms a, b with compact support in $X - \partial X$ such that $d(a + db) = da = \phi_0 - R$. Then $d_1(\chi A + a) = \phi$ and surjectivity is proved.

We now construct the map j_1. Recall that (cf. HLZ; §1])

$$H^k(X, \partial X; S^1) \cong H^k_{\rm cpt}(X - \partial X; S^1) \cong \frac{\{f \in \mathcal{F}^k_{\rm cpt}(X - \partial X) : df \in \mathcal{R}^{k+1}_{\rm cpt}(X - \partial X)\}}{d\mathcal{F}^{k-1}_{\rm cpt}(X - \partial X) + \mathcal{R}^k_{\rm cpt}(X - \partial X)}$$

Choose $f \in \mathcal{F}^k_{\rm cpt}(X - \partial X)$ with $df = R \in \mathcal{R}^{k+1}_{\rm cpt}(X - \partial X)$, and write $f = a + db$ where a and b are $L^1_{\rm loc}$-forms with compact support in $X - \partial X$ (cf. Lemma 2.4). Then $a \in \mathcal{S}^k(X, \partial X)$ and we set $j_1(f) \equiv \langle a \rangle \in \widehat{I\!H}^k(X, \partial X)$. Note that if $f = a' + db'$ is another such decomposition, then $a - a' = d(c' - c)$ and $\langle a \rangle = \langle a' \rangle$. Clearly $j_1 = 0$ on $d\mathcal{F}^{k-1}_{\rm cpt}(X - \partial X) + \mathcal{R}^k_{\rm cpt}(X - \partial X) = d\mathcal{E}^{k-1}_{\rm cpt}(X - \partial X) + \mathcal{R}^k_{\rm cpt}(X - \partial X)$, and so it descends to the quotient $H^k(X, \partial X; S^1)$.

To see that j_1 is injective, let $f = a + db$ as above and suppose $a = dc + S \in \mathcal{T}^k(X, \partial X)$ where c is smooth and zero on ∂X. By Lemma 2.3 there exists an $L^1_{\rm loc}$-form e, smooth near ∂X, such that $c_0 = c - de \equiv 0$ near ∂X. Then $a = dc_0 + S \equiv 0$ in $H^k(X, \partial X; S^1)$.

We now prove the exactness of (A) in the middle. Suppose $a \in \mathcal{S}^k(X, \partial X)$ and $\delta_1(\langle a \rangle) = 0$. Then $da = -R \in \mathcal{R}^{k+1}_{\rm cpt}(X - \partial X)$. Thus, in a neighborhood N of ∂X we have that a is smooth, $da = 0$ and $a\big|_{\partial X} = 0$. By Lemma 2.3 there exists $b \in \mathcal{E}^{k-1}(N)$ with $db = a$ and $b\big|_{\partial X} = 0$. Then $\widetilde{a} = a - d(\chi b)$, with χ as above, is equivalent to a in $\widehat{I\!H}^k(X, \partial X)$. Since $\widetilde{a}$ has compact support in $X - \partial X$ and $d\widetilde{a} = -R$, we see that $\langle \widetilde{a} \rangle$ lies in the image of j_1.

We now prove the surjectivity of δ_2. Fix $u \in H^{k+1}(X, \partial X; \mathbb{Z})$ and choose a cycle $R \in u$. Then there is a smooth form $\phi \in \mathcal{Z}^{k+1}_0(X, \partial X)$ such that $\phi - R = df$ for $f \in \mathcal{F}^k_{\rm cpt}(X - \partial X)$. By Lemma 2.4 $f = a + db$ where a is $L^1_{\rm loc}$ with compact support in $X - \partial X$. Then $a \in \mathcal{S}^k(X, \partial X)$ and $\delta_2(\langle a \rangle) = u$.

Now consider an element $a \in \mathcal{S}^k(X, \partial X)$ with $\delta_2(\langle a \rangle) = 0$. Then $da = \phi - R$ where ϕ is smooth and $R = dS$ for some $S \in \mathcal{R}^k_{\mathrm{cpt}}(X - \partial X)$. Then $\widetilde{a} = a - S \equiv a$ in $\widehat{I\!\!H}^k(X, \partial X)$ and $d\widetilde{a} = 0$ on X. Since $\widetilde{a}$ is smooth near ∂X, standard de Rham theory shows that there is an L^1_{loc}-form b with compact support in $X - \partial X$ such that $\widetilde{a} - db$ is smooth. Hence, $\langle a \rangle = \langle \widetilde{a} \rangle \in \widehat{I\!\!H}^k_\infty(X, \partial X)$. $\square$

Note that

$$\ker(\delta_1) \cap \ker(\delta_2) \;\cong\; \frac{H^k(X, \partial X; \mathbb{R})}{H^k(X, \partial X; \mathbb{Z})_{\mathrm{free}}} \;\cong\; \frac{H^k_{\mathrm{cpt}}(X - \partial X; \mathbb{R})}{H^k_{\mathrm{cpt}}(X - \partial X; \mathbb{Z})} \tag{2.5}$$

§**3. The star product.** In this section we prove the following.

Theorem 3.1. *There are functorial bilinear mappings*

$$\begin{aligned} \widehat{I\!\!H}^k(X, \partial X) \times \widehat{I\!\!H}^\ell(X, \partial X) &\xrightarrow{*} \widehat{I\!\!H}^{k+\ell+1}(X, \partial X) \qquad \textit{and} \\ \widehat{I\!\!H}^k(X, \partial X) \times \widehat{I\!\!H}^\ell(X) &\xrightarrow{*} \widehat{I\!\!H}^{k+\ell+1}(X, \partial X) \end{aligned}$$

which make $\widehat{I\!\!H}^k(X, \partial X)$ a graded commutative ring and $\widehat{I\!\!H}^*(X)$ a graded $\widehat{I\!\!H}^k(X, \partial X)$-module. With this structure the maps δ_1, δ_2 are ring and module homomorphisms.

Proof. Fix $\alpha \in \widehat{I\!\!H}^k(X, \partial X)$ and $\beta \in \widehat{I\!\!H}^\ell(X)$. Then from [HLZ] we know that there exist sparks $a \in \alpha$ and $b \in \beta$ with

$$da \;=\; \phi - R \qquad \text{and} \qquad db \;=\; \psi - S$$

with $\phi \in \mathcal{Z}_0^{k+1}(X, \partial X), \psi \in \mathcal{Z}_0^{\ell+1}(X), R \in Z^{k+1}_{\mathrm{rect}}(X, \partial X)$ and $S \in Z^{\ell+1}_{\mathrm{rect}}(X)$, so that the wedge-intersection products $R \wedge b$ and $R \wedge S$ are well defined. Furthermore, if supp $S \subset\subset X - \partial X$ we can also assume that $a \wedge S$ is well defined. We then define

$$a * b \;=\; a \wedge \psi + (-1)^{k+1} R \wedge b, \tag{3.2}$$

and if $S \in Z^{\ell+1}_{\mathrm{rect}}(X, \partial X)$ or if $a \in \mathcal{E}^k_{\mathrm{cpt}}(X - \partial X)$, we can also define

$$a \widetilde{*} b \;=\; a \wedge S + (-1)^{k+1} \phi \wedge b. \tag{3.3}$$

Since a is smooth near ∂X and $a\big|_{\partial X} = 0$, $a * b$ also has these properties (as well as $a \widetilde{*} b$ when it is defined). Note that

$$d(a * b) \;=\; d(a \widetilde{*} b) \;=\; \phi \wedge \psi - R \wedge S \tag{3.4}$$

The arguments from [HLZ] easily adapt to show that $\langle a * b \rangle$ depends only on $\langle a \rangle$ and $\langle b \rangle$, and that $\langle a * b \rangle = \langle a \widetilde{*} b \rangle$ (when it is defined). Associativity, commutativity, etc. are straightforward. Equation (3.4) establishes the homomorphism propertes of δ_1 and δ_2. $\square$

§4. Smooth Pontrjagin Duals. The exact sequences of Proposition 2.2 show that $\widehat{I\!H}^*(X, \partial X)$ has a natural topology making it a topological group (in fact a topological ring) for which δ_1 and d_2 are continuous homomorphisms. Essentially it is a product of the standard C^∞-topology on forms with the standard topology on the torus $H^k(X, \partial X; \mathbb{R})/H^k(X, \partial X; \mathbb{Z})$. It can also be defined as the quotient of the topology induced on sparks by the embedding $\mathcal{S}^k(X, \partial X) \subset \mathcal{F}^k(X) \times \mathcal{E}^{k+1}(X, \partial X) \times \mathcal{R}^{k+1}_{\rm cpt}(X - \partial X)$ sending $a \mapsto (a, d_1 a, d_2 a)$. (Similar remarks apply to $\widehat{I\!H}^*(X)$.)

It is natual to consider the dual to $\widehat{I\!H}^*(X, \partial X)$ in the sense of Pontrjagin. For an abelian topological group A we denote by $A^\star \equiv \mathrm{Hom}_{\rm cont}(A, S^1)$ the group of continuous homomorphisms $h : A \to S^1$. Then 2.2(B) gives a dual sequence

$$0 \to H^{k+1}(X, \partial X; \mathbb{Z})^\star \to \widehat{I\!H}^k(X, \partial X)^\star \xrightarrow{\rho} \widehat{I\!H}^k_\infty(X, \partial X)^\star \to 0. \tag{4.1}$$

where ρ is the restriction mapping.

Definition 4.2. An element $f \in \widehat{I\!H}^k_\infty(X, \partial X)^\star$ is called **smooth** if there exists a form $\omega \in \mathcal{Z}^{n-k}_0(X)$ such that

$$f(\alpha) \ \equiv \ \int_X a \wedge \omega \ \ (\mathrm{mod} \ \ \mathbb{Z})$$

for $a \in \alpha \in \widehat{I\!H}^k_\infty(X, \partial X)$. An element $f \in \widehat{I\!H}^k(X, \partial X)^\star$ is called **smooth** if $\rho(f)$ is smooth. The set of these is called the **smooth Pontrjagin dual** of $\widehat{I\!H}^k(X, \partial X)$ and is denoted by $\widehat{I\!H}^k(X, \partial X)^{\star\infty} = \mathrm{Hom}_\infty(\widehat{I\!H}^k(X, \partial X), S^1)$

Proposition 4.3. *The smooth Pontrjagin dual* $\widehat{I\!H}^k(X, \partial X)^{\star\infty}$ *is dense in* $\widehat{I\!H}^k(X, \partial X)^\star$.

Proof. Applying δ_1 to $\widehat{I\!H}^k_\infty(X, \partial X)$ gives an exact sequence

$$0 \to T \to \widehat{I\!H}^k_\infty(X, \partial X) \to d\mathcal{E}^k(X, \partial X) \to 0$$

where $T = H^k(X, \partial X; \mathbb{R})/H^k_{\rm free}(X, \partial X; \mathbb{Z})$, with dual sequence

$$0 \to d\mathcal{E}^k(X, \partial X)^\star \to \widehat{I\!H}^k_\infty(X, \partial X)^\star \to T^\star \to 0 \tag{4.4}$$

Observe that $T^\star = H^k_{\rm free}(X, \partial X; \mathbb{Z}) \cong H^{n-k}_{\rm free}(X; \mathbb{Z})$, and that $d\mathcal{E}^k(X, \partial X)^\star = \{d\mathcal{E}^k(X, \partial X)\}'$ (the topological vector space dual) which is exactly the space of currents of degree $n-k-1$ on X restricted to the closed subspace $d\mathcal{E}^k(X, \partial X)$. This gives a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & d\mathcal{E}^{n-k-1}(X) & \longrightarrow & \mathcal{Z}^{n-k}_0(X) & \longrightarrow & H^{n-k}_{\rm free}(X; \mathbb{Z}) & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow \cong & & \\
0 & \longrightarrow & d\mathcal{D}'^{n-k-1}(X) & \longrightarrow & \widehat{I\!H}^k_\infty(X, \partial X)^\star & \longrightarrow & T^\star & \longrightarrow & 0
\end{array}$$

with exact rows. Since $\mathcal{E}^{n-k-1}(X)$ is dense in $\mathcal{D}'^{n-k-1}(X)$, the result follows. $\square$

There is a parallel story for $\widehat{I\!H}^*(X)$. The analogue of 2.2(B) gives an exact sequence

$$0 \to H^{k+1}(X; \mathbb{Z})^\star \to \widehat{I\!H}^k(X)^\star \xrightarrow{\rho} \widehat{I\!H}^k_\infty(X)^\star. \to 0. \tag{4.5}$$

Definition 4.6. An element $f \in \widehat{I\!H}^k_\infty(X)^\star$ is called **smooth** if there exists a form $\omega \in \mathcal{Z}_0^{n-k}(X, \partial X)$ such that

$$f(\alpha) \;\equiv\; \int_X a \wedge \omega \pmod{\mathbb{Z}}$$

for $a \in \alpha \in \widehat{I\!H}^k_\infty(X)$ An element $f \in \widehat{I\!H}^k(X)^\star$ is called **smooth** if $\rho(f)$ is smooth. The set of these is called the **smooth Pontrjagin dual** of $\widehat{I\!H}^k(X)$ and is denoted $\widehat{I\!H}^k(X)^{\star\infty} = \mathrm{Hom}_\infty(\widehat{I\!H}^k(X), S^1)$

Proposition 4.7. *The smooth Pontrjagin dual $\widehat{I\!H}^k(X)^{\star\infty}$ is dense in $\widehat{I\!H}^k(X)^\star$.*

Proof. Applying δ_1 to $\widehat{I\!H}^k_\infty(X)$ gives an exact sequence

$$0 \to T \to \widehat{I\!H}^k_\infty(X) \to d\mathcal{E}^k(X) \to 0,$$

where $T = H^k(X; \mathbb{R})/H^k(X; \mathbb{Z})$, with dual sequence

$$0 \to d\mathcal{E}^k(X)^\star \to \widehat{I\!H}^k_\infty(X)^\star \to T^\star \to 0. \tag{4.8}$$

Observe now that $T^\star = H^k(X; \mathbb{Z}) \cong H^{n-k}(X, \partial X; \mathbb{Z})$, and $d\mathcal{E}^k(X)^\star = \{d\mathcal{E}^k(X)\}'$ is the space of currents of degree $n-k-1$ on X restricted to the closed subspace $d\mathcal{E}^k(X)$. This gives a commutative diagram:

$$\begin{tikzcd}
\mathcal{E}^{n-k-1}(X, \partial X) \arrow[r, "d"] \arrow[d] & \mathcal{Z}_0^{n-k}(X, \partial X) \arrow[r] \arrow[d] & H^{n-k}(X, \partial X; \mathbb{Z}) \arrow[r] \arrow[d, "\cong"] & 0 \\
\mathcal{D}'^{n-k-1}(X) \arrow[r, "d"] & \widehat{I\!H}^k_\infty(X)^\star \arrow[r] & T^\star \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Since $\mathcal{E}^{n-k-1}(X, \partial X)$ is dense in $\mathcal{D}'^{n-k-1}(X)$, the result follows. $\square$

§5. Lefschetz-Pontrjagin Duality. This brings us to the main result of the paper.

Theorem 5.1. *Let X be a compact, oriented n-manifold with boundary ∂X. Then the biadditive mapping*

$$\widehat{I\!H}^k(X, \partial X) \times \widehat{I\!H}^{n-k-1}(X) \longrightarrow S^1$$

given by

$$(\alpha, \beta) \quad \mapsto \quad (\alpha * \beta)\,[X]$$

induces isomorphisms

$$\mathcal{D} : \widehat{I\!H}^k(X, \partial X) \xrightarrow{\cong} \widehat{I\!H}^{n-k-1}(X)^{\star\infty} \qquad \textit{and}$$
$$\mathcal{D}' : \widehat{I\!H}^k(X) \xrightarrow{\cong} \widehat{I\!H}^{n-k-1}(X, \partial X)^{\star\infty}$$

Proof. Fix $\alpha \in \widehat{I\!H}^k(X, \partial X)$ and suppose $(\alpha * \beta)[X] = 0$ for all $\beta \in \widehat{I\!H}^{n-k-1}(X)$. We shall show that $\alpha = 0$. Choose a spark $a \in \alpha$ and write $da = \phi - R$ as in 1.4. Then for all smooth forms $b \in \mathcal{E}^{n-k-1}(X)$ we have by (3.3) that

$$\alpha * \langle b \rangle\ [X] \ = \ (-1)^{k+1} \int_X \phi \wedge b \ \equiv \ 0 \mod \mathbb{Z}$$

since $d_2 b = 0$. It follows that $\phi = 0$.

Hence, $da = -R \in \mathcal{R}^{k+1}_{\text{cpt}}(X - \partial X)$ is a cycle with $[R] \in H^{k+1}_{\text{cpt}}(X - \partial X; \mathbb{Z})_{\text{tor}} \cong H_{n-k-1}(X - \partial X; \mathbb{Z})_{\text{tor}}$. Choose any $u \in H^{n-k}(X; \mathbb{Z})_{\text{tor}} \cong H_k(X, \partial X; \mathbb{Z})_{\text{tor}}$, and choose a relative cycle $S \in u$. Let m be the order of u. Then there is a $(k+1)$-chain T on X with $dT = mS$ rel ∂X. Set $b = -\frac{1}{m}T$ and consider b as a spark of degree $n-k-1$ on X with $db = -S$. Now we may assume S and T to have been chosen so that supp $(S) \cap$supp $(R) = \emptyset$ and T meets R properly. Then

$$\begin{aligned} 0 \ = \ \alpha * \langle b \rangle\ [X] \ &\equiv \ (-1)^{k+1} R \wedge b\,[X] \mod \mathbb{Z} \\ &\equiv \ (-1)^{k+1} \frac{1}{m} R \wedge T\,[X] \mod \mathbb{Z} \\ &\equiv \ (-1)^{k+1} \text{Lk}([R], [S]) \mod \mathbb{Z} \\ &\equiv \ (-1)^{k+1} \text{Lk}(\delta_2 \alpha, u) \mod \mathbb{Z} \end{aligned}$$

where Lk denotes the de Rham-Seifert linking between the groups $H_{n-k-1}(X - \partial X; \mathbb{Z})_{\text{tor}}$ and $H_k(X, \partial X; \mathbb{Z})_{\text{tor}}$. By the non-degeneracy of this pairing we conclude that $\delta_2 \alpha = 0$.

Therefore $\alpha \in \ker(\delta_1) \cap \ker(\delta_2)$ can be represented by a smooth d-closed form $a \in \mathcal{E}^k(X, \partial X)$. In fact by Lemma 2.3 we may choose a to have compact support in $X - \partial X$. Now for any cycle $S \in Z^{n-k}_{\text{rect}}(X)$, i.e., any k-dimensional rectifiable current $S \in \mathcal{R}_k(X)$ with $dS \in \mathcal{R}_{k-1}(\partial X)$, we can find $\psi \in \mathcal{E}^{n-k}(X)$ and $b \in \mathcal{E}^{n-k-1}_{L^1_{\text{loc}}}(X)$ with $db = \psi - S$. Then by (3.3) we have that

$$\begin{aligned} 0 \ = \ \alpha * \langle b \rangle\ [X] \ &\equiv \ a \wedge S\ [X] \mod \mathbb{Z} \\ &\equiv \ \int_S a \mod \mathbb{Z}. \end{aligned}$$

Hence, a represents the zero class in $\text{Hom}(H_k(X, \partial X; \mathbb{Z}), \mathbb{R})/\text{Hom}(H_k(X, \partial X; \mathbb{Z}), \mathbb{Z}) \cong H^k(X, \partial X; \mathbb{R})/H^k(X, \partial X; \mathbb{Z})_{\text{free}}$, and by (2.2) and (2.5) we conclude that $\alpha = 0$. Thus

the map $\mathcal{D}$ is injective.

To see that $\mathcal{D}$ is surjective consider the commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^k(X,\partial X;\, S^1) & \xrightarrow{j_1} & \widehat{I\!\!H}^k(X,\partial X) & \xrightarrow{\delta_1} & \mathcal{Z}_0^{k+1}(X,\partial X) & \longrightarrow & 0 \\
 & & \cong\big\downarrow & & \big\downarrow\mathcal{D} & & \big\downarrow\mathcal{D}_0 & & \\
0 & \longrightarrow & \mathrm{Hom}(H^{n-k}(X;\,\mathbb{Z}), S^1) & \longrightarrow & \widehat{I\!\!H}^{n-k-1}(X)^\star & \xrightarrow{\rho} & \widehat{I\!\!H}_\infty^{n-k-1}(X)^\star & \longrightarrow & 0
\end{array}$$

where the top row is 2.2(A) and the bottom row is the dual of 2.2(B). By definition $\mathcal{D}_0$ is onto the smooth elements in $\widehat{I\!\!H}^{n-k-1}(X)^\star$ and therefore the map $\mathcal{D}$ is surjective.

The proof that $\mathcal{D}'$ is an isomorphism is parallel. Fix $\beta \in \widehat{I\!\!H}^{n-k-1}(X)$ and suppose $(\alpha * \beta)[X] = 0$ for all $\alpha \in \widehat{I\!\!H}^k(X, \partial X)$. We shall show that $\beta = 0$. Choose a spark $b \in \beta$ and write $db = \psi - S$ as in 1.4. Then for all smooth forms $a \in \mathcal{E}^k(X, \partial X)$ we have by (3.3) that

$$\langle a\rangle * \beta\,[X] \;=\; \int_X a \wedge \psi \;\equiv\; 0 \quad \mathrm{mod}\ \mathbb{Z}$$

since $d_2 a = 0$. It follows that $\psi = 0$.

Hence, $db = -S \in \mathcal{R}^{n-k}(X)$ is a relative cycle with torsion homology class $[S] \in H^{n-k}(X;\,\mathbb{Z})_{\mathrm{tor}} \cong H_k(X, \partial X;\,\mathbb{Z})_{\mathrm{tor}}$. Choose $u \in H^{k+1}(X, \partial X;\,\mathbb{Z})_{\mathrm{tor}} \cong H_{n-k-1}(X;\,\mathbb{Z})_{\mathrm{tor}}$, and choose a cycle $R \in u$ with support in $X - \partial X$. Let m be the order of u. Then there is a $(n-k-1)$-chain T in $X - \partial X$ with $dT = mR$. Set $a = -\frac{1}{m}T$ and consider a as a spark of degree k on X with $da = -R$. Now we may assume R and T to have been chosen so that $\mathrm{supp}\,(R) \cap \mathrm{supp}\,(S) = \emptyset$ and T meets S properly. Then

$$\begin{aligned}
0 \;=\; \langle a\rangle * \beta\,[X] \;&\equiv\; (-1)^{k+1} a \wedge S\,[X] \ \ \mathrm{mod}\ \mathbb{Z} \\
&\equiv\; (-1)^{k+1}\frac{1}{m} T \wedge S\,[X] \ \ \mathrm{mod}\ \mathbb{Z} \\
&\equiv\; (-1)^{k+1}\mathrm{Lk}([R],[S]) \ \ \mathrm{mod}\ \mathbb{Z} \\
&\equiv\; (-1)^{k+1}\mathrm{Lk}(u, \delta_2\beta) \ \ \mathrm{mod}\ \mathbb{Z}
\end{aligned}$$

where Lk denotes the de Rham-Seifert linking as before. We conclude that $\delta_2\alpha = 0$.

Therefore $\beta \in \ker(\delta_1) \cap \ker(\delta_2)$ can be represented by a smooth d-closed form $b \in \mathcal{E}^{n-k-1}(X)$. Now for any cycle $R \in Z^{k+1}_{\mathrm{rect}}(X, \partial X)$, i.e., any k-dimensional rectifiable current $R \in \mathcal{R}_{n-k-1}(X - \partial X)$ with $dR = 0$, we can find $\phi \in \mathcal{E}^{k+1}(X, \partial X)$ and $a \in \mathcal{E}^k_{L^1_{\mathrm{loc}}}(X, \partial X)$ with $da = \phi - R$. Then by (3.2) we have that

$$\begin{aligned}
0 \;=\; \langle a\rangle * \beta\,[X] \;&\equiv\; (-1)^{k+1} R \wedge b\ [X] \ \ \mathrm{mod}\ \mathbb{Z} \\
&\equiv\; (-1)^{n(k+1)} \int_R b \ \ \mathrm{mod}\ \mathbb{Z}.
\end{aligned}$$

Hence, b represents the zero class in $\mathrm{Hom}(H_{n-k-1}(X;\,\mathbb{Z}), \mathbb{R})/\mathrm{Hom}(H_{n-k-1}(X;\,\mathbb{Z}), \mathbb{Z}) \cong H^{n-k-1}(X;\,\mathbb{R})/H^{n-k-1}(X;\,\mathbb{Z})_{\mathrm{free}}$, and by (2.2) and (2.5) we conclude that $\beta = 0$. Thus the map $\mathcal{D}'$ is injective.

The surjectivity of $\mathcal{D}'$ follows as before from the commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & H^{n-k-1}(X;\, S^1) \arrow[r, "j_1"] \arrow[d, "\cong"'] & \widehat{I\!\!H}^{n-k-1}(X) \arrow[r, "\delta_1"] \arrow[d, "\mathcal{D}"] & \mathcal{Z}_0^{n-k}(X) \arrow[r] \arrow[d, "\mathcal{D}_0"] & 0 \\
0 \arrow[r] & \mathrm{Hom}(H^{k+1}(X, \partial X;\, \mathbb{Z}), S^1) \arrow[r] & \widehat{I\!\!H}^{k}(X, \partial X)^\star \arrow[r, "\rho"] & \widehat{I\!\!H}^{k}_\infty(X, \partial X)^\star \arrow[r] & 0.
\end{tikzcd}$$

This completes the proof. $\square$

§**6. Coboundary maps.** It is natural to ask if there is a coboundary mapping ∂ with the property that the sequence

$$(6.1)\quad \ldots \to \widehat{I\!\!H}^{k-1}(\partial X) \xrightarrow{\partial} \widehat{I\!\!H}^{k}(X, \partial X) \xrightarrow{j} \widehat{I\!\!H}^{k}(X) \xrightarrow{\rho} \widehat{I\!\!H}^{k}(\partial X) \xrightarrow{\partial} \widehat{I\!\!H}^{k+1}(X, \partial X) \to \ldots$$

is exact. The differential-form-component of characters makes this impossible. However, there do exist natural coboundary maps ∂ with the following properties:

(1) Under δ_2 the sequence (6.1) becomes the standard long exact sequence in integral cohomology.

(2) Under δ_1 the sequence (6.1) becomes a sequence of smooth d-closed forms which induces the standard long exact sequence in real cohomology.

Recall that the definitions of Thom maps and Gysin maps for differential characters depend essentially on a choice of "normal geometry". This will also be true for our coboundary maps. Fix a tubular neighborhood N_0 of ∂X in X and an identification $N_0 \cong \partial X \times [0, 2)$, and let $\pi : N_0 \to \partial X$ be the projection. Set $N = \partial X \times [0, 1) \subset N_0$ and let $\mathbb{I}_N$ be the characteristic function of this subset. Let χ be a smooth approximation to $\mathbb{I}_N$; specifically choose $\chi(x, t) = \chi(t)$ where $\chi \equiv 1$ near 0 and $\chi(t) = 0$ for $t \geq 1$. Then set

$$\lambda \;\equiv\; \chi - \mathbb{I}_N \in \widehat{I\!\!H}^{0}(X)$$

Note that $d\lambda = d\chi - [\partial N]$ has compact support in $X - \partial X$.

Definition 6.2. We define the coboundary map $\partial = \partial_\lambda : \widehat{I\!\!H}^{k}(\partial X) \to \widehat{I\!\!H}^{k+1}(X, \partial X)$ by

$$\partial(a) \;=\; \pi^* a * \lambda.$$

Verification of (1) and (2) above is straightforward, and the details are omitted.

§**7. Sequences and duality.** At the level of cohomology the long exact sequences for the pair $(X, \partial X)$ are related by the duality mappings. There is an analogous diagram for differential characters:

$$\begin{tikzcd}
\widehat{I\!\!H}^{k}(X, \partial X) \arrow[r, "j"] \arrow[d, "\mathcal{D}"'] & \widehat{I\!\!H}^{k}(X) \arrow[r, "\rho"] \arrow[d, "\mathcal{D}"'] & \widehat{I\!\!H}^{k}(\partial X) \arrow[r, "\partial"] \arrow[d, "\mathcal{D}"'] & \widehat{I\!\!H}^{k+1}(X, \partial X) \arrow[d, "\mathcal{D}"'] \\
\widehat{I\!\!H}^{n-k-1}(X)^\star \arrow[r, "j^*"] & \widehat{I\!\!H}^{n-k-1}(X, \partial X)^\star \arrow[r, "\partial^*"] & \widehat{I\!\!H}^{n-k-2}(\partial X)^\star \arrow[r, "\rho^*"] & \widehat{I\!\!H}^{n-k-2}(X)^\star
\end{tikzcd}$$

and it is natural to ask whether this diagram commutes (up to sign). The square on the left is evidently commutative. The other two squares commute up to an error term which we now analyse.

We begin with the square on the right. Fix $\alpha \in \widehat{I\!\!H}^k(\partial X)$ and $\beta \in \widehat{I\!\!H}^{n-k-2}(X)$ and choose L^1_{loc}-sparks $a_0 \in \alpha$ and $b \in \beta$ with $da_0 = \phi_0 - R_0$ and $db = \psi - S$ as usual. Let $a = \pi^* a_0$, $\phi = \pi^*\phi_0$ and $R = \pi^* R_0$ denote the pull-backs to the collar neighborhood of ∂X via the projection $\pi : N_0 \to \partial X$ defined in §6. Then

$$\{(\mathcal{D} \circ \partial)(\alpha)\}(\beta) \;=\; (\pi^* a * b * \lambda)[X] \;=\; \{(a * b) \wedge d\chi + (-1)^n d_2(a * b)\lambda\}[X]. \tag{7.1}$$

Now we may assume that $S\big|_{N_0} = \pi^* S_0$ for some $S_0 \in \mathcal{R}^{k+1}(\partial X)$, and we may further assume that supp $(R_0) \cap$ supp $(S_0) = \emptyset$ because $\dim(R_0) + \dim(S_0) = n - 2$. Hence, $d_2(a * b) = \pi^* R_0 \wedge \pi^* S_0 = \pi^*(R_0 \wedge S_0) = 0$, and from (7.1) we see that

$$\begin{aligned} (-1)^{n-1}\{(\mathcal{D} \circ \partial)(\alpha)\}(\beta) \;&=\; (-1)^{n-1}(a * b) \wedge d\chi[X] \\ &=\; (a * b)[\partial X] - \chi d(a * b)[X] \\ &=\; \{(\rho^* \circ \mathcal{D})(\alpha)\}(\beta) - \chi d(a * b)[X]. \end{aligned}$$

Now $d(a * b) = \phi \wedge \psi - R \wedge S = \phi \wedge \psi$ and we can write $\psi = \psi_1 + dt \wedge \psi_2$ as in the proof of Lemma 2.3. Since $\phi = \pi^*\phi_0$ we see that $\phi \wedge \psi_1 = 0$ and we conclude that

$$\begin{aligned} \{(\rho^* \circ \mathcal{D})(\alpha)\}(\beta) + (-1)^n\{(\mathcal{D} \circ \partial)(\alpha)\}(\beta) \;&=\; \int_N \phi \wedge \chi dt \wedge \psi_2 \\ &=\; \int_{\partial X} \phi \wedge \int_0^1 \chi(t)\, dt \wedge \psi_2 \\ &=\; \int_{\partial X} \phi \wedge \pi_* \{\chi(t)\, dt \wedge \psi_2\} \equiv E(\lambda). \end{aligned} \tag{7.2}$$

Thus for example we see that $(\rho^* \circ \mathcal{D})(\alpha) = (-1)^{n-1}(\mathcal{D} \circ \partial)(\alpha)$ on all β which are π^*-pull backs in N. Furthermore, we can consider the family of sparks $\lambda_\epsilon \equiv r_\epsilon^* \lambda$ where $r_\epsilon : \partial X \times [0, \epsilon) \to \partial X[0, 1)$ is given by $r_\epsilon(x, t) = (x, t/\epsilon)$. From (7.2) we see that

$$\lim_{\epsilon \to 0} E(\lambda_\epsilon) = 0.$$

A similar analysis applies to middle square in the diagram and we have the following.

Proposition 7.3. *The duality diagram above commutes in the limit as $\epsilon \to 0$.*

This is the best one can expect. The "commutators" in this diagram do not lie in the smooth dual. Of course by Propositions 4.3 and 4.7 they do lie in its closure.

Here is an explicit example of this non-commutativity. Let $X = S^2 \times D^3$ be the product of the 2-sphere and the 3-disk. Choose sparks $\alpha \in \mathcal{S}^1(S^2)$ and $b \in \mathcal{S}^2(D^3)$ with $da = \omega - [x_0]$ and $db = \Omega - [0]$ for some $x_0 \in S^2$, where ω and Ω are unit volume forms on S^2 and D^3 respectively. Direct calculation shows that

$$(a * b)[\partial X] \;=\; 1 \qquad \text{but} \qquad (a * \lambda * b)[X] \;=\; \int_{D^3} (1 - \chi)\Omega \;<\; 1.$$

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